

The Direct and Converse Inequalities for Jackson-Type Operators on Spherical Cap *

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Abstract

Approximation on the spherical cap is different from that on the sphere which requires us to construct new operators. This paper discusses the approximation on the spherical cap. That is, so called Jackson-type operator $\{J_{k,s}^m\}_{k=1}^\infty$ is constructed to approximate the function defined on the spherical cap $D(x_0, \gamma)$. We thus establish the direct and inverse inequalities and obtain saturation theorems for $\{J_{k,s}^m\}_{k=1}^\infty$ on the cap $D(x_0, \gamma)$. Using methods of K -functional and multiplier, we obtain the inequality

$$C_1 \|J_{k,s}^m(f) - f\|_{D,p} \leq \omega^2(f, k^{-1})_{D,p} \leq C_2 \max_{v \geq k} \|J_{v,s}^m(f) - f\|_{D,p}$$

and that the saturation order of these operators is $O(k^{-2})$, where $\omega^2(f, t)_{D,p}$ is the modulus of smoothness of degree 2, the constants C_1 and C_2 are independent of k and f .

MSC(2000): 41A17

Key words: approximation; saturation order; Jackson-type operator; spherical cap

*The research was supported by the National Natural Science Foundation of China (No. 60873206) and the Natural Science Foundation of Zhejiang Province of China (No. Y7080235)

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1 Introduction

In the past decades, many mathematicians dedicated to establish the Jackson and Bernstein-type theorems on the sphere. Early works, such as Butzer [3], Nikol'skiĭ [14] and [15], Lizorkin [6] had successfully established the direct and inverse theorems on the sphere. In early 1990s, Li and Yang [5] constructed Jackson operators on the sphere and obtained the Jackson and Bernstein-type theorems for the Jackson operators.

Jackson operator on the sphere is defined by (see [6], [5])

$$J_{k,s}(f)(x) := \frac{1}{|\mathbb{S}^{n-2}|} \int_{\mathbb{S}^{n-1}} f(y) \mathcal{D}_{k,s}(\arccos x \cdot y) f(y) d\omega(y) \quad (1.1)$$

where k and s are positive integers,

$$\mathcal{D}_{k,s}(\theta) = \left(\frac{\sin \frac{k\theta}{2}}{\sin \frac{\theta}{2}} \right)^{2s}$$

is the classical Jackson kernel, f is measurable function of degree p on the sphere \mathbb{S}^{n-1} in \mathbb{R}^n , $d\omega(y)$ is the elementary surface piece, $|\mathbb{S}^{n-1}|$ is the measurement of \mathbb{S}^{n-1} . For $f \in L^p(\mathbb{S}^{n-1})$, ($1 \leq p \leq \infty$) ($L^\infty(\mathbb{S}^{n-1})$ is the collection of continuous functions on \mathbb{S}^{n-1}), Li and Yang [5] proved that

$$C_1 \|J_{k,s}(f) - f\|_{\mathbb{S}^{n-1},p} \leq \omega^2(f, k^{-1}) \leq C_2 k^{-2} \sum_{v=1}^k v \|J_{v,s}(f) - f\|_{\mathbb{S}^{n-1},p}$$

and the saturation order for $J_{k,s}$ is k^{-2} , where C_1 and C_2 are independent of positive integer k and f , and $\omega^2(f, t)$ is the modulus of smoothness of degree 2 on the unit sphere \mathbb{S}^{n-1} .

On the spherical caps, we desire to obtain the same results. The main difficulty we face is to establish the Bernstein-type inequality (for polynomials) on the cap. Recently, Belinsky, Dai and Ditzian [1] constructed m -th translation operator S_θ^m when discussing the averages of functions on the sphere. This inspires us and allows us to construct the m -th Jackson-type operator $J_{k,s}^m$ on the spherical cap. Fortunately, the Bernstein-type inequality holds for $J_{k,s}^m$, which helps us get the direct and inverse inequalities of approximation on the spherical cap. Finally, we obtain that the saturation order for the constructed Jackson-type operator is k^{-2} , the same to that of the Jackson operator on the sphere.

2 Definitions and Auxiliary Notations

Throughout out this paper, we denote by the letters C and C_i (i is either positive integers or variables on which C depends only) positive constants depending only on the dimension n . Their value may be different at different occurrences, even within the same formula. We shall denote the points in \mathbb{S}^{n-1} by $x, x_0, x_1, \dots, y, y_0, y_1, \dots$, and z, z_0, z_1, \dots , and the elementary surface piece on \mathbb{S}^{n-1} by $d\omega$. If it is necessary, we shall write $d\omega(x)$ referring to the variable of the integration. The notation $a \approx b$ means that there exists a positive constant C such that $C^{-1}b \leq a \leq Cb$ where C is independent of some variable n on which a and b both depend.

Next, we introduce some concepts and properties of sphere as well as caps (see [8], [12]). The volume of \mathbb{S}^{n-1} is

$$\Omega_{n-1} := \int_{\mathbb{S}^{n-1}} d\omega = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Corresponding to $d\omega$, the inner product on \mathbb{S}^{n-1} is defined by

$$\langle f, g \rangle := \int_{\mathbb{S}^{n-1}} f(x)\overline{g(x)}d\omega(x).$$

Denote by $L^p(\mathbb{S}^{n-1})$ the space of p -integrable functions on \mathbb{S}^{n-1} endowed with the norms

$$\|f\|_\infty := \|f\|_{L^\infty(\mathbb{S}^{n-1})} := \operatorname{ess\,sup}_{x \in \mathbb{S}^{n-1}} |f(x)|$$

and

$$\|f\|_p := \|f\|_{L^p(\mathbb{S}^{n-1})} := \left\{ \int_{\mathbb{S}^{n-1}} |f(x)|^p d\omega(x) \right\}^{1/p} < \infty, \quad 1 \leq p < \infty.$$

We denote by $D(x_0, \gamma)$ the spherical cap with center x_0 and angle $0 < \gamma \leq \frac{\pi}{2}$, i.e.,

$$D(x_0, \gamma) := \{x \in \mathbb{S}^{n-1} : x \cdot x_0 \geq \cos \gamma\},$$

and by $D(\gamma)$ the volume of $D(x_0, \gamma)$, i.e.,

$$D(\gamma) := \int_0^\gamma |\mathbb{S}^{n-2}| \sin^{n-2} \theta d\theta.$$

Then for fixed x_0 and γ , $L^p(D(x_0, \gamma))$ is a Banach space endowed with the norm $\|\cdot\|_{D,p}$ defined by

$$\|f\|_{D,\infty} := \operatorname{ess\,sup}_{x \in D(x_0, \gamma)} |f(x)|$$

and

$$\|f\|_{D,p} := \left\{ \int_{D(x_0,\gamma)} |f(x)|^p d\omega(x) \right\}^{1/p}, \quad 1 \leq p < \infty.$$

For any $f \in L^p(D(x_0, \gamma))$, we note

$$f^*(x) = \begin{cases} f(x), & x \in D(x_0, \gamma), \\ 0, & x \in \mathbb{S}^{n-1} \setminus D(x_0, \gamma), \end{cases}$$

and clearly, $f^* \in L^p(\mathbb{S}^{n-1})$ and $\|f^*\|_p = \|f\|_{D,p}$. This allows us to introduce some operators on spherical cap using existing operators on the sphere.

Definition 2.1 Suppose $T : L^p(\mathbb{S}^{n-1}) \rightarrow L^p(\mathbb{S}^{n-1})$ is an operator on \mathbb{S}^{n-1} , then

$$T_{x_0,\gamma} : L^p(D(x_0, \gamma)) \rightarrow L^p(D(x_0, \gamma))$$

$$T_{x_0,\gamma}(f)(x) = T(f^*)(x), \quad x \in D(x_0, \gamma)$$

is called the operator on $D(x_0, \gamma)$ introduced by T . We may use the notation T instead of $T_{x_0,\gamma}$ for convenience if without mixing up.

We now make a brief introduction of projection operators $Y_j(\cdot)$ by ultraspherical (Gegenbauer) polynomials $\{G_j^\lambda\}_{j=1}^\infty$ ($\lambda = \frac{n-2}{2}$) for discussion of saturation property of Jackson operators.

Ultraspherical polynomials $\{G_j^\lambda\}_{j=1}^\infty$ are defined in terms of the generating function (see [10]):

$$\frac{1}{(1 - 2tr + r^2)^\lambda} = \sum_{j=0}^{\infty} G_j^\lambda(t) r^j$$

where $|r| < 1$, $|t| \leq 1$.

For any $\lambda > 0$, we have (see [10])

$$G_1^\lambda(t) = 2\lambda t \tag{2.2}$$

and

$$\frac{d}{dt} G_j^\lambda(t) = 2\lambda G_{j-1}^{\lambda+1}(t). \tag{2.3}$$

When $\lambda = \frac{n-2}{2}$ (see [12]),

$$G_j^\lambda(t) = \frac{\Gamma(2\lambda + j)}{\Gamma(j+1)\Gamma(2\lambda)} P_j^n(t), \quad j = 0, 1, 2, \dots$$

where $P_j^n(t)$ is the Legendre polynomial of degree j . Particularly,

$$G_j^\lambda(1) = \frac{\Gamma(2\lambda + j)}{\Gamma(j + 1)\Gamma(2\lambda)} P_j^n(1) = \frac{\Gamma(2\lambda + j)}{\Gamma(j + 1)\Gamma(2\lambda)}, \quad j = 0, 1, 2, \dots$$

Therefore,

$$P_j^n(t) = \frac{G_j^\lambda(t)}{G_j^\lambda(1)}. \quad (2.4)$$

Besides, for any $j = 0, 1, 2, \dots$, and $|t| \leq 1$, $|P_j^n(t)| \leq 1$ (see [8]).

The projection operators is defined by

$$Y_j(f)(x) = \frac{\Gamma(\lambda)(n + \lambda)}{2\pi^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} G_j^\lambda(x \cdot y) f^*(y) d\omega(y).$$

It follows from (2.2), (2.3) and (2.4) that

$$\lim_{t \rightarrow 1^-} \frac{1 - (P_j^n(t))^m}{1 - (P_1^n(t))^m} = \frac{j(j + 2\lambda)}{2\lambda + 1}. \quad (2.5)$$

In the same way, we define the inner product on $D(x_0, \gamma)$ as follows,

$$\langle f, g \rangle_D := \int_{D(x_0, \gamma)} f(x) \overline{g(x)} d\omega(x).$$

We denote by $\tilde{\Delta}$ the Laplace-Beltrami operator

$$\tilde{\Delta} f := \sum_{i=1}^n \frac{\partial^2 g(x)}{\partial x_i^2} \Big|_{|x|=1}, \quad g(x) = f\left(\frac{x}{|x|}\right),$$

by which we define a K -function on $D(x_0, \gamma)$ as

$$K(f, \delta)_{D,p} := \inf \left\{ \|f - g\|_{D,p} + \delta \|\tilde{\Delta} g\|_{D,p} : g, \tilde{\Delta} g \in L^p(D(x_0, \gamma)) \right\}.$$

For $f \in L^1(D(x_0, \gamma))$, the translation operator is defined by

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{n-2}| \sin^{n-2} \theta} \int_{x \cdot y = \cos \theta} f^*(y) d\omega'(y),$$

where $d\omega'(y)$ denotes the the elementary surface piece on the sphere $\{y \in D(x_0, \gamma) : x \cdot y = \cos \theta\}$. Then we have

$$\int_{D(x_0, \gamma)} f(x) d\omega(x) = \int_0^\pi S_\theta(f)(x_0) |\mathbb{S}^{n-2}| \sin^{n-2} \theta d\theta.$$

The modulus of smoothness of f is defined by

$$\omega^2(f, \delta)_{D,p} := \sup_{0 < \theta \leq \delta} \|S_\theta(f) - f\|_{D,p}.$$

Using the method of [3], we have

$$C_1\omega^2(f, \delta)_{D,p} \leq K(f, \delta^2)_{D,p} \leq C_2\omega^2(f, \delta)_{D,p}. \quad (2.6)$$

We introduce m -th translation operator in terms of multipliers (see [7], [9], [12])

$$S_\theta^m(f) = \sum_{j=0}^{\infty} \left(\frac{G_j^\lambda(\cos \theta)}{G_j^\lambda(1)} \right)^m Y_j(f) \equiv \sum_{j=0}^{\infty} (P_j^n(\cos \theta))^m Y_j(f), \quad f \in L^p(D(x_0, \gamma)). \quad (2.7)$$

It has been proved that (see [12])

$$\begin{aligned} S_\theta(f)(x) &= \frac{1}{|\mathbb{S}^{n-2}| \sin^{n-2} \theta} \int_{x \cdot y = \cos \theta} f^*(y) d\omega'(y) = \sum_{j=0}^{\infty} P_j^n(\cos \theta) Y_j(f)(x) \\ &= S_\theta^1(f)(x) \end{aligned} \quad (2.8)$$

With the help of S_θ^m , we can construct Jackson-type operators on $D(x_0, \gamma)$.

Definition 2.2 For $f \in L^p(D(x_0, \gamma))$, the m -th Jackson-type operator of degree k on $D(x_0, \gamma)$ is defined by

$$J_{k,s}^m(f)(x) = \int_0^\gamma S_\theta^m(f)(x) \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta,$$

where $\lambda = \frac{n-2}{2}$, and $\tilde{D}_{k,s}(\theta) = A_{\gamma,k,s}^{-1} \left(\frac{\sin^{2s} \frac{k\theta}{2}}{\sin^{2s-1} \frac{\theta}{2}} \right)$ satisfying $\int_0^\gamma \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta = 1$.

Remark 2.1 We may notice that $\tilde{D}_{k,s}(\theta)$ is a bit different from classical Jackson kernel

$$\mathcal{D}_{k,s}(\theta) = \left(\frac{\sin \frac{k\theta}{2}}{\sin \frac{\theta}{2}} \right)^{2s}.$$

This difference will help us to prove Bernstein inequality for $J_{k,s}^m$. For sake of ensuring that Bernstein inequality for $J_{k,s}^m$ holds, γ has to be no more than $\frac{\pi}{2}$. Particularly, for $m = 1$, we have

$$\begin{aligned} J_{k,s}^1(f)(x) &= \int_0^\gamma S_\theta^1(f)(x) \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \\ &= \int_0^\gamma \frac{1}{|\mathbb{S}^{n-2}| \sin^{2\lambda} \theta} \int_{x \cdot y = \cos \theta} f^*(y) d\omega'(y) \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \\ &= \frac{1}{|\mathbb{S}^{n-2}|} \int_{D(x_0, \gamma)} f^*(y) \tilde{D}_{k,s}(\arctan(x \cdot y)) d\omega(y) \\ &= \frac{1}{|\mathbb{S}^{n-2}|} \int_{D(x_0, \gamma) \cap D(x, \gamma)} f(y) \tilde{D}_{k,s}(\arctan(x \cdot y)) d\omega(y). \end{aligned}$$

Finally, we introduce the definition of saturation for operators (see [2]).

Definition 2.3 Let $\varphi(\rho)$ be a positive function with respect to ρ , $0 < \rho < \infty$, tending monotonely to zero as $\rho \rightarrow \infty$. For $\rho > 0$, I_ρ is a sequence of operators. If there exists $\mathcal{K} \subseteq L^p(D(x_0, \gamma))$ such that:

- (i) If $\|I_\rho(f) - f\|_{D,p} = o(\varphi(\rho))$, then $I_\rho(f) = f$;
- (ii) $\|I_\rho(f) - f\|_{D,p} = O(\varphi(\rho))$ if and only if $f \in \mathcal{K}$;

then I_ρ is said to be saturated on $L^p(D(x_0, \gamma))$ with order $O(\varphi(\rho))$ and \mathcal{K} is called its saturation class.

3 Some Lemmas

In this section, we show some lemmas on both S_θ^m and $J_{k,s}^m$ as the preparation for the main results. For S_θ^m , we have

Lemma 3.1 For $f \in L^p(D(x_0, \gamma))$, $1 \leq p \leq \infty$, $0 < \theta \leq \pi$,

- (i) For $1 \leq m_1 < m$, $S_\theta^m = S_\theta^{m_1} S_\theta^{m-m_1}$;
- (ii) For $m \geq 1$, $\|S_\theta^m(f)\|_{D,p} \leq \|f\|_{D,p}$;
- (iii) For $m \geq 1$, $\|S_\theta^m(f) - f\|_{D,p} \leq m \|S_\theta(f) - f\|_{D,p}$;
- (iv) For $m > \frac{2(\lfloor \frac{n}{2} \rfloor + 3)}{n-2}$, $0 < \theta \leq \frac{\pi}{2}$, $\|\tilde{\Delta} S_\theta^m(f)\|_{D,p} \leq C_m \theta^{-2} \|f\|_{D,p}$,
where $C_m \rightarrow 0$, as $m \rightarrow \infty$;
- (v) For $m \geq 1$, and f which satisfies $\tilde{\Delta} f \in L^p(D(x_0, \gamma))$, $\|\tilde{\Delta} S_\theta^m(f)\|_{D,p} \leq \|\tilde{\Delta} f\|_{D,p}$.

Proof. (i), (ii) and (iii) are clear. Using Remark 3.5 of [1], we can obtain (iv). For (v), we have

$$\tilde{\Delta} S_\theta(f) = - \sum_{j=0}^{\infty} j(j+2\lambda) P_j^n(\cos \theta) Y_j(f) = S_\theta(\tilde{\Delta} f),$$

which implies

$$\|\tilde{\Delta} S_\theta^m(f)\|_{D,p} = \|S_\theta(\tilde{\Delta} S_\theta^{m-1}(f))\|_{D,p} \leq \|\tilde{\Delta} S_\theta^{m-1}(f)\|_{D,p} \leq \cdots \leq \|\tilde{\Delta} f\|_{D,p}. \quad \square$$

We need the following lemma.

Lemma 3.2 For $\beta \geq -2$, $2s > \beta + 2\lambda + 1$, $0 < \gamma \leq \pi$, $n \geq 3$, and $k \geq 1$, we have

$$\int_0^\gamma \theta^\beta \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \approx k^{-\beta}, \quad (3.9)$$

where $\lambda = \frac{n-2}{2}$.

Proof. A simple calculation gives, for $\beta \geq -2$,

$$\int_0^\gamma \theta^\beta \frac{\sin^{2s} \frac{k\theta}{2}}{\sin^{2s-1} \frac{\theta}{2}} \sin^{2\lambda} \theta d\theta \leq C \left(\int_0^\infty \frac{\sin^{2s}}{\theta^{2s-(2\lambda+\beta+1)}} d\theta \right) k^{2s-(2\lambda+\beta+2)},$$

$$\int_0^\gamma \theta^\beta \frac{\sin^{2s} \frac{k\theta}{2}}{\sin^{2s-1} \frac{\theta}{2}} \sin^{2\lambda} \theta d\theta \geq C \left(\int_0^{\frac{k\gamma}{4}} \frac{\sin^{2s}}{\theta^{2s-(2\lambda+\beta+1)}} d\theta \right) k^{2s-(2\lambda+\beta+2)},$$

$$\int_0^\gamma \theta^\beta \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta = \frac{\int_0^\gamma \theta^\beta \frac{\sin^{2s} \frac{k\theta}{2}}{\sin^{2s-1} \frac{\theta}{2}} \sin^{2\lambda} \theta d\theta}{\int_0^\gamma \frac{\sin^{2s} \frac{k\theta}{2}}{\sin^{2s-1} \frac{\theta}{2}} \sin^{2\lambda} \theta d\theta} \approx k^{-\beta}. \quad \square$$

For Jackson-type operators, we have

Lemma 3.3 For $f \in L^p(D(x_0, \gamma))$, $1 \leq p < \infty$,

- (i) $\|J_{k,s}^m(f)\|_{D,p} \leq \|f\|_{D,p}$;
- (ii) For f which satisfies $\tilde{\Delta}f \in L^p(D(x_0, \gamma))$, $\|\tilde{\Delta}J_{k,s}^m(f)\|_{D,p} \leq \|\tilde{\Delta}f\|_{D,p}$;
- (iii) For $n \geq 3$, $m > \frac{2([\frac{n}{2}]+3)}{n-2}$ and $0 < \gamma \leq \frac{\pi}{2}$, $\|\tilde{\Delta}J_{k,s}^m(f)\|_{D,p} \leq C_m k^2 \|f\|_{D,p}$.

Proof. From the definition and (ii) and (v) of Lemma 3.1, (i) and (ii) are clear. We just have to add the proof of (iii). In fact, using Minkowski inequality, (iv) of Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} \|\tilde{\Delta}J_{k,s}^m(f)\|_{D,p} &\leq \int_0^\gamma \|\tilde{\Delta}S_\theta^m(f)\|_{D,p} \tilde{D}_{k,s}(\theta) \sin^2 \theta d\theta \\ &\leq C_m \|f\|_{D,p} \int_0^\gamma \theta^{-2} \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \\ &\approx C_{m,\gamma} k^2 \|f\|_{D,p}, \end{aligned}$$

where the constant in the approximation is independent of m and k . \square

The following lemma is useful in the proof of Bernstein-type inequality for Jackson-type operators.

Lemma 3.4 ([13]) *Suppose that for nonnegative sequences $\{\sigma_k\}_{k=1}^{\infty}$, $\{\tau_k\}_{k=1}^{\infty}$ with $\sigma_1 = 0$ the inequality*

$$\sigma_n \leq \left(\frac{k}{n}\right)^p \sigma_k + \tau_k, \quad p > 0, \quad 1 \leq k \leq n$$

is satisfied for any positive integer n . Then one has

$$\sigma_n \leq C_p n^{-p} \sum_{k=1}^n k^{p-1} \tau_k. \quad (3.10)$$

The following lemma gives the multiplier representation of $J_{k,s}^m(f)$, which follows from Definition 2.2 and (2.7).

Lemma 3.5 *For $f \in L^p(D(x_0, \gamma))$, $J_{k,s}^m(f)$ has the representation*

$$J_{k,s}^m(f)(x) = \sum_{j=0}^{\infty} \xi_k^m(j) Y_j(f)(x) \quad (3.11)$$

where

$$\xi_k^m(j) = \int_0^\gamma \tilde{D}_{k,s}(\theta) (P_j^n(\cos \theta))^m \sin^{2\lambda} \theta \, d\theta, \quad j = 0, 1, 2, \dots$$

The following lemma is useful for determining the saturation order. It can be deduced by the methods of [2] and [4].

Lemma 3.6 *Suppose that $\{I_\rho\}_{\rho>0}$ is a sequence of operators on $L^p(D(x_0, \gamma))$, and there exists function series $\{\lambda_\rho(j)\}_{j=1}^{\infty}$ with respect to ρ , such that*

$$I_\rho(f)(x) = \sum_{j=0}^{\infty} \lambda_\rho(j) Y_j(f)(x)$$

for every $f \in L^p(D(x_0, \gamma))$. If for any $j = 0, 1, 2, \dots$, there exists $\varphi(\rho) \rightarrow 0+$ ($\rho \rightarrow \rho_0$) such that

$$\lim_{\rho \rightarrow \rho_0} \frac{1 - \lambda_\rho(j)}{\varphi(\rho)} = \tau_j \neq 0,$$

then $\{I_\rho\}_{\rho>0}$ is saturated on $L^p(D(x_0, \gamma))$ with the order $O(\varphi(\rho))$ and the collection of all constants is the invariant class for $\{I_\rho\}_{\rho>0}$ on $L^p(D(x_0, \gamma))$.

4 Main Results and Their Proofs

In this section, we shall discuss the main results, that is, the lower and upper bounds as well as the saturation order for Jackson-type operators on $L^p(D(x_0, \gamma))$.

The following theorem gives the Jackson-type inequality for $J_{k,s}^m$.

Theorem 4.1 *For any integer $m \geq 1$ and $0 < \gamma \leq \frac{\pi}{2}$, $\{J_{k,s}^m\}_{k=1}^\infty$ is the series of Jackson-type operators on $L^p(D(x_0, \gamma))$ defined above, and $g \in L_p^{(2)}(D(x_0, \gamma)) := \{f \in L^p(D(x_0, \gamma)) : \tilde{\Delta}f \in L^p(D(x_0, \gamma))\}$, $1 \leq p \leq \infty$.*

Then

$$\|J_{k,s}^m(g) - g\|_{D,p} \leq C_{m,\gamma} k^{-2} \|\tilde{\Delta}g\|_{D,p}.$$

Therefore, for $f \in L^p(D(x_0, \gamma))$,

$$\|J_{k,s}^m(f) - f\|_{D,p} \leq C_{m,\gamma} \omega^2(f, k^{-1})_{D,p}, \quad (4.12)$$

where C is independent of k and f .

Proof. Since $g \in L_p^{(2)}(D(x_0, \gamma))$, we have (see [2])

$$S_\theta(g)(x) - g(x) = \int_0^\theta \sin^{2-n} \nu \int_0^\nu \sin^{n-2} \tau S_\tau(\tilde{\Delta}g)(x) d\tau d\nu. \quad (4.13)$$

and it is true that

$$\sup_{\theta>0} \left\{ \theta^{-2} \int_0^\theta \sin^{2-n} \nu \int_0^\nu \sin^{n-2} \tau d\tau d\nu \right\} < \infty.$$

Therefore (explained below),

$$\begin{aligned} \|J_{k,s}^m(g) - g\|_{D,p} &= \left\| \int_0^\gamma \tilde{D}_{k,s}(\theta) (S_\theta^m(g)(x) - g(x)) \sin^{n-2} \theta d\theta \right\|_{D,p} \\ &\leq m \int_0^\gamma \tilde{D}_{k,s}(\theta) \sin^{n-2} \theta \left(\int_0^\theta \sin^{2-n} \nu \int_0^\nu \sin^{n-2} \tau \|S_\tau(\tilde{\Delta}g)\|_{D,p} d\tau d\nu \right) d\theta \\ &\leq m \sup_{\theta>0} \left\{ \theta^{-2} \int_0^\theta \sin^{2-n} \nu \int_0^\nu \sin^{n-2} \tau d\tau d\nu \right\} \|\tilde{\Delta}g\|_{D,p} \int_0^\gamma \theta^2 \tilde{D}_{k,s}(\theta) \sin^{n-2} \theta d\theta \\ &\leq C_{m,\gamma} k^{-2} \|\tilde{\Delta}g\|_{D,p}, \end{aligned}$$

where the Minkowski inequality, (4.13) and Lemma 3.1 are used in the first inequality, and the second one and third one is deduced from (3.9) and Lemma 3.1. From (2.6) and (i) of Lemma 3.3, it is easy to deduce (4.12). \square

Next, we prove the Bernstein-type inequality for $J_{k,s}^m(f)(x)$ for $f \in L^p(D(x_0, \gamma))$.

Theorem 4.2 Assume $\{J_{k,s}^m\}_{k=1}^\infty$, $m > \frac{2(\lfloor \frac{n}{2} \rfloor + 3)}{n-2}$ are m -th Jackson-type operators on $D(x_0, \gamma)$. For $f \in L^p(D(x_0, \gamma))$, $0 < \gamma \leq \frac{\pi}{2}$, then there exists a constant C independent of k and f such that

$$\omega^2(f, k^{-1})_{D,p} \leq C \max_{v \geq k} \|J_{v,s}^m(f) - f\|_{D,p} \quad (4.14)$$

holds for every $f \in L^p(D(x_0, \gamma))$ and every integer k .

Proof. Li and Yang [5] have proved the Marchaud-Stečkin inequality for Jackson operator on the sphere. Following the method in [5], we first prove the Marchaud-Stečkin inequality for $J_{k,s}^m$:

$$\omega^2(f, k^{-1})_{D,p} \leq C_1 k^{-2} \sum_{v=1}^k v \|J_{v,s}^m(f) - f\|_{D,p}. \quad (4.15)$$

Let

$$\sigma_k = k^{-2} \|\tilde{\Delta} J_{k,s}^m(f)\|_{D,p}, \quad \tau_k = \|J_{k,s}^m(f) - f\|_{D,p},$$

then for $v = 1, 2, \dots, k$, by Lemma 3.3,

$$\begin{aligned} \sigma_k &= k^{-2} \|\tilde{\Delta} J_{k,s}^m(f) - \tilde{\Delta} J_{k,s}^m(J_{v,s}^m(f)) + \tilde{\Delta} J_{k,s}^m(J_{v,s}^m(f))\|_{D,p} \\ &= k^{-2} \|\tilde{\Delta} J_{k,s}^m(f - J_{v,s}^m(f))\|_{D,p} + \|\tilde{\Delta} J_{k,s}^m(J_{v,s}^m(f))\|_{D,p} \\ &\leq k^{-2} \left(C k^2 \|f - J_{v,s}^m(f)\|_{D,p} + \|\tilde{\Delta} J_{v,s}^m(f)\|_{D,p} \right) \\ &= C \|f - J_{v,s}^m(f)\|_{D,p} + k^{-2} \|\tilde{\Delta} J_{v,s}^m(f)\|_{D,p} \\ &= C \tau_v + \left(\frac{v}{k}\right)^2 \sigma_v, \end{aligned}$$

so we can deduce from Lemma 3.4 (where p is set to be 2) that

$$\sigma_k \leq C k^{-2} \sum_{v=1}^k v \tau_v$$

that is,

$$\|\tilde{\Delta} J_{k,s}^m(f)\|_{D,p} \leq C \sum_{v=1}^k v \|J_{v,s}^m(f) - f\|_{D,p}. \quad (4.16)$$

Since there exists $\frac{k}{2} \leq r \leq k$ such that

$$\|J_{r,s}^m(f) - f\|_{D,p} = \min_{\frac{k}{2} \leq v \leq k} \|J_{v,s}^m(f) - f\|_{D,p}$$

then,

$$\begin{aligned}
K(f, k^{-2})_{D,p} &\leq \|J_{r,s}^m(f) - f\|_{D,p} + k^{-2} \|\tilde{\Delta} J_{r,s}^m(f)\|_{D,p} \\
&\leq 4k^{-2} \sum_{\frac{k}{2} \leq v \leq k} v \|J_{v,s}^m(f) - f\|_{D,p} + Ck^{-2} \sum_{v=1}^k v \|J_{v,s}^m(f) - f\|_{D,p} \\
&\leq Ck^{-2} \sum_{v=1}^k v \|J_{v,s}^m(f) - f\|_{D,p}.
\end{aligned}$$

By (2.6), we obtain that

$$\begin{aligned}
\omega^2(f, k^{-1})_{D,p} &\leq CK(f, k^{-2})_{D,p} \\
&\leq Ck^{-2} \sum_{v=1}^k v \|J_{v,s}^m(f) - f\|_{D,p}.
\end{aligned}$$

So (4.15) holds, and it implies that

$$\omega^2(f, k^{-1})_{D,p} \leq C_1 k^{-1-\frac{1}{2}} \sum_{v=1}^k v^{\frac{1}{2}} \|J_{v,s}^m(f) - f\|_{D,p}. \quad (4.17)$$

In order to prove (4.14), we have to show that

$$\begin{aligned}
\omega^2(f, k^{-1})_{D,p} &\approx \frac{1}{k^2} \max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p} \\
&\approx \frac{1}{k^{2+\frac{1}{4}}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p}.
\end{aligned} \quad (4.18)$$

We first prove

$$\omega^2(f, k^{-1})_{D,p} \approx \frac{1}{k^2} \max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p}. \quad (4.19)$$

It follows from (4.12) and (4.17) that

$$\begin{aligned}
\omega^2(f, k^{-1})_{D,p} &\leq C_1 k^{-2} k^{\frac{1}{2}} \sum_{v=1}^k v^{-\frac{3}{2}} \left(v^2 \|J_{v,s}^m(f) - f\|_{D,p} \right) \\
&\leq C_1 k^{-2} \left(\max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p} \right) \left(k^{\frac{1}{2}} \sum_{v=1}^k v^{-\frac{3}{2}} \right) \\
&\leq C_3 k^{-2} \max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p} \\
&\leq C_4 k^{-2} \max_{1 \leq v \leq k} v^2 \omega^2(f, v^{-1})_{D,p} \\
&\leq C_4 \left(k^{-2} \max_{1 \leq v \leq k} v^2 \left(1 + \left(\frac{k}{v} \right)^2 \right) \right) \omega^2(f, k^{-1})_{D,p} \\
&\leq 2C_4 \omega^2(f, k^{-1})_{D,p}.
\end{aligned}$$

Then we prove

$$\omega^2(f, k^{-1})_{D,p} \approx \frac{1}{k^{2+\frac{1}{4}}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p}.$$

In fact, the proof is similar to that of (4.19).

$$\begin{aligned} \omega^2(f, k^{-1})_{D,p} &\leq C_1 k^{-2-\frac{1}{4}} k^{\frac{3}{4}} \sum_{k=1}^k v^{-\frac{7}{4}} \left(v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \right) \\ &\leq C_1 k^{-2-\frac{1}{4}} \left(\max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \right) \left(k^{\frac{3}{4}} \sum_{v=1}^k v^{-\frac{7}{4}} \right) \\ &\leq C_5 k^{-2-\frac{1}{4}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \\ &\leq C_6 k^{-2-\frac{1}{4}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \omega^2(f, v^{-1})_{D,p} \\ &\leq C_6 \left(k^{-2-\frac{1}{4}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \left(1 + \left(\frac{k}{v} \right)^2 \right) \right) \omega^2(f, k^{-1})_{D,p} \\ &\leq 2C_6 \omega^2(f, k^{-1})_{D,p}. \end{aligned}$$

Hence,

$$\omega^2(f, k^{-1})_{D,p} \approx \frac{1}{k^2} \max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p} \approx \frac{1}{k^{2+\frac{1}{4}}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p}.$$

Now we can complete the proof of (4.14). Let

$$\max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} = k_1^{2+\frac{1}{4}} \|J_{k_1,s}^m(f) - f\|_{D,p}, \quad 1 \leq k_1 \leq k.$$

it follows from (4.18) that

$$\begin{aligned} k^{-2} k_1^2 \|J_{k_1,s}^m(f) - f\|_{D,p} &\leq k^{-2} \max_{1 \leq v \leq k} v^2 \|J_{v,s}^m(f) - f\|_{D,p} \\ &\leq \frac{C_7}{k^{2+\frac{1}{4}}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \\ &= \frac{C_7}{k^{2+\frac{1}{4}}} k_1^{2+\frac{1}{4}} \|J_{k_1,s}^m(f) - f\|_{D,p}. \end{aligned}$$

Thus, $C_7^{-4} k \leq k_1$. Since $k_1 \leq k$, then $k \approx k_1$, we obtain from (4.18) that

$$\begin{aligned} \omega^2(f, k^{-1})_{D,p} &\leq C_8 k^{-2-\frac{1}{4}} \max_{1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \\ &= C_8 k^{-2-\frac{1}{4}} \left(k_1^{2+\frac{1}{4}} \|J_{k_1,s}^m(f) - f\|_{D,p} \right) \\ &\leq C_8 k^{-2-\frac{1}{4}} \max_{k_1 \leq v \leq k} v^{2+\frac{1}{4}} \|J_{v,s}^m(f) - f\|_{D,p} \\ &\leq C_8 \max_{k_1 \leq v \leq k} \|J_{v,s}^m(f) - f\|_{D,p}. \end{aligned}$$

Noticing that $k \approx k_1$, we may rewrite the above inequality as

$$\omega^2(f, k^{-1})_{D,p} \leq C_9 \max_{v \geq k} \|J_{v,s}^m(f) - f\|_{D,p}.$$

This completes the proof. \square

We thus obtain the corollary of Theorem 4.1 and Theorem 4.2 that:

Corollary 4.1 *Suppose that $\{J_{k,s}^m\}_{k=1}^\infty$, $m > \frac{2(\lfloor \frac{n}{2} \rfloor + 3)}{n-2}$, are Jackson-type operators on the spherical cap $D(x_0, \gamma)$, $0 < \gamma \leq \frac{\pi}{2}$, then the following are equivalent for any $f \in L^p(D(x_0, \gamma))$, $\alpha > 0$,*

- (i) $\|J_{k,s}^m(f) - f\|_{D,p} = O(k^{-\alpha})$, $k \rightarrow \infty$;
- (ii) $\omega^2(f, \delta)_{D,p} = O(\delta^\alpha)$, $\delta \rightarrow 0+$. \square

Theorem 4.3 *Suppose that $\{J_{k,s}^m\}_{k=1}^\infty$, $m \geq 1$ are Jackson-type operators on the spherical cap $D(x_0, \gamma)$, $0 < \gamma \leq \frac{\pi}{2}$. Then $\{J_{k,s}^m\}_{k=1}^\infty$ are saturated on $L^p(D(x_0, \gamma))$ with order k^{-2} and the collection of constants is their invariant class.*

Proof. We obtain from Lemma 3.2 that, for $v = 1, 2, \dots$,

$$\begin{aligned} 1 - \xi_k^m(1) &= \int_0^\gamma \tilde{D}_{k,s}(\theta) \left(1 - \left(\frac{G_1^\lambda(\cos \theta)}{G_1^\lambda(1)} \right)^m \right) \sin^{2\lambda} \theta \, d\theta \approx \int_0^\gamma \tilde{D}_{k,s}(\theta) \sin^2 \frac{\theta}{2} \sin^{2\lambda} \theta \, d\theta \\ &\approx k^{-2}. \end{aligned}$$

By Lemma 3.6, if it is true that for $j = 0, 1, 2, \dots$,

$$\lim_{k \rightarrow \infty} \frac{1 - \xi_k^m(j)}{1 - \xi_k^m(1)} = \frac{j(j+2\lambda)}{2\lambda+1},$$

then the proof is completed.

In fact, for any $0 < \delta < \gamma$, it follows from (3.9) that

$$\begin{aligned} \int_\delta^\gamma \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta \, d\theta &\leq \int_\delta^\gamma \left(\frac{\theta}{\delta} \right)^3 \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta \, d\theta \\ &\leq \delta^{-3} \int_0^\gamma \theta^3 \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta \, d\theta \leq C_{\delta,s} k^{-3}. \end{aligned}$$

We deduce from (2.5) that, for any $\varepsilon > 0$, there exists $\delta > 0$, for $0 < \theta < \delta$, it holds that

$$\left| (1 - (P_j^n(\cos \theta))^m) - \frac{j(j+2\lambda)}{2\lambda+1} (1 - (P_1^n(\cos \theta))^m) \right| \leq \varepsilon (1 - (P_1^n(\cos \theta))^m).$$

Then it follows that

$$\begin{aligned}
& \left| (1 - \xi_k^m(j)) - \frac{j(j+2\lambda)}{2\lambda+1} (1 - \xi_k^m(1)) \right| \\
= & \left| \int_0^\gamma \tilde{D}_{k,s}(\theta) (1 - (P_j^n(\cos \theta))^m) \sin^{2\lambda} \theta d\theta \right. \\
& \left. - \int_0^\gamma \tilde{D}_{k,s}(\theta) (1 - (P_1^n(\cos \theta))^m) \frac{j(j+2\lambda)}{2\lambda+1} \sin^{2\lambda} \theta d\theta \right| \\
= & \left| \int_0^\gamma \tilde{D}_{k,s}(\theta) \left((1 - (P_j^n(\cos \theta))^m) - (1 - (P_1^n(\cos \theta))^m) \frac{j(j+2\lambda)}{2\lambda+1} \right) \sin^{2\lambda} \theta d\theta \right| \\
\leq & \int_0^\delta \tilde{D}_{k,s}(\theta) \varepsilon \sin^{2\lambda} \theta d\theta + 2 \int_\delta^\gamma \tilde{D}_{k,s}(\theta) \sin^{2\lambda} \theta \left(1 + \frac{j(j+2\lambda)}{2\lambda+1} \right) d\theta \\
\leq & \varepsilon k^{-2} + C_{\delta,s} k^{-3}.
\end{aligned}$$

So,

$$\lim_{k \rightarrow \infty} \frac{1 - \xi_k(j)}{1 - \xi_k(1)} = \frac{j(j+2\lambda)}{2\lambda} \neq 0.$$

Therefore, we obtain by Lemma 3.6 that the saturation order for $J_{k,s}^m$ is $O(1 - \xi_k(1)) \approx k^{-2}$. \square

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