Approximation by Boolean Sums of Jackson Operators on the Sphere *

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Abstract

This paper concerns the approximation by the Boolean sums of Jackson operators $\oplus^r J_{k,s}(f)$ on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. We prove the following the direct and inverse theorem for $\oplus^r J_{k,s}(f)$: there are constants C_1 and C_2 such that

$$C_1\| \oplus^r J_{k,s}f - f\|_p \leq \omega^{2r}(f, k^{-1})_p \leq C_2 \max_{v\geq k} \| \oplus^r J_{k,s}f - f\|_p$$

for any positive integer k and any pth Lebesgue integrable functions f defined on $\mathbb{S}^{n-1}$, where $\omega^{2r}(f,t)_p$ is the modulus of smoothness of degree $2r$ of f. We also prove that the saturation order for $\oplus^r J_{k,s}$ is k^{-2r}.

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1 Introduction

In past decades, many mathematicians have dedicated to establish the Jackson and Bernstein-type theorems on the sphere. Lizorkin and Nikol'skiĭ [7] constructed Boolean sums of Jackson operators $\oplus^r J_{k,s}$ (which will be defined in the next section) for proving the direct and inverse theorems on a special Banach space $H_p^r(\mathbb{S}^{n-1})$. Later, by using a modulus of smoothness as metric, Lizorkin and Nikol'skiĭ [8] obtained the direct estimate for Jackson operators (i.e., the 1-th Boolean sums of Jackson operators) approximating continuous function defined on the unit sphere $\mathbb{S}^{n-1}$. In 1991, Li and Yang [6] used the equivalent relation between the K-functional and the smoothness (see [2]) and established an inverse inequality of weak type, which was also called Steckin-Marchaud type inequality. Recently, Ditzian [5] proved an equivalence relation between K-functional and modulus of smoothness of high order, which will provide a tool and allow us to make a proof for direct and inverse theorems for the Boolean sums of Jackson operators.

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Actually, after improving a Steckin-Marchaud type inequality, we will prove the direct and inverse theorem of approximation for arbitrary r-th Boolean sums of Jackson operators $\oplus^r J_{k,s}$ approximating p-th Lebesgue integrable function on the sphere. Particularly, a converse inequality of strong type (see [4]) for $\oplus^r J_{k,s}$ will be established. Moreover, we will use the method of multipliers and obtain the saturation order of $\oplus^r J_{k,s}$.

2 Definitions and Auxiliary Notations

Let $\mathbb{S}^{n-1}$ be the unit sphere in Euclidean space $\mathbb{R}^n$. We denote by the letters C and C_i positive constants, where i is either positive integers or variables on which C depends only. Their values may be different at different occurrences, even within the same formula. We shall denote by x and y the points of $\mathbb{S}^{n-1}$. The notation $a \approx b$ means that there exists a positive constant C such that $C^{-1}b \le a \le Cb$.

We now introduce some concepts and properties of sphere (see also [9], [**?**]). The volume of $\mathbb{S}^{n-1}$ is

$$\left|\mathbb{S}^{n-1}\right| := \int_{\mathbb{S}^{n-1}} d\omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

Denote by $L^p(\mathbb{S}^{n-1})$ the space of p-th integrable functions on $\mathbb{S}^{n-1}$ endowed with the norms

$$\|f\|_\infty := \|f\|_{L^\infty(\mathbb{S}^{n-1})} := \operatorname*{ess\ sup}_{x\in\mathbb{S}^{n-1}} |f(x)|$$

and

$$\|f\|_p := \|f\|_{L^p(\mathbb{S}^{n-1})} := \left\{\int_{\mathbb{S}^{n-1}} |f(x)|^p d\omega(x)\right\}^{1/p} < \infty, \quad 1 \le p < \infty.$$

For $f \in L^1(\mathbb{S}^{n-1})$, the translation operator is defined by (see for instance [**?**])

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{n-2}|\sin^{n-2}\theta} \int_{x\cdot y=\cos\theta} f(y)d\omega'(y), \quad 0 < \theta < \pi$$

where $d\omega'(y)$ denotes the elementary surface piece on $\mathbb{S}^{n-2}$.
We set

$$S_\theta^{(0)}(f) := f,\ S_\theta^{(j)}(f) := S_\theta S_\theta^{(j)}(f), \quad j = 1, 2, 3, \dots$$

and introduce the spherical differences (see [7])

$$\Delta_\theta^1(f) := \Delta_\theta(f) := S_\theta(f) - f$$

and

$$\Delta_\theta^r(f) := \Delta_\theta\Delta_\theta^{r-1}(f) = \sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j} S_\theta^{(j)}(f) = (S_\theta - I)^r f, \quad r = 2, 3, \dots$$

where I is the identity operator on $L^p(\mathbb{S}^{n-1})$. Then the modulus of smoothness of degree $2r$ of $f \in L^p(\mathbb{S}^{n-1})$ is defined by (see [11])

$$\omega^{2r}(f, t) := \sup_{0<\theta\le t} \|\Delta_\theta^r f\|_p, \quad 0 < t < \pi,\ r = 1, 2, \dots.$$

We denote by $\widetilde{\Delta}$ the Laplace-Beltrami operator

$$\widetilde{\Delta} f := \sum_{i=1}^{n} \frac{\partial^2 g(x)}{\partial x_i^2}\bigg|_{|x|=1}, \quad g(x) := f\left(\frac{x}{|x|}\right), \quad \left(f,\ \widetilde{\Delta} f \in L^p(\mathbb{S}^{n-1})\right)$$

by which we introduce K-functional of degree $2r$ on $\mathbb{S}^{n-1}$ as

$$K_{2r}(f, \widetilde{\Delta}, t^{2r})_p := \inf\left\{\|f-g\|_p + t^{2r}\|\widetilde{\Delta}^r g\|_p :\ g,\ \widetilde{\Delta}^r g \in L^p(\mathbb{S}^{n-1})\right\}.$$

For the modulus of smoothness and K-functional, Ditzian [5] has obtained the following equivalence relation

$$\omega^{2r}(f,t)_p \approx K_{2r}(f, \widetilde{\Delta}, t^{2r})_p. \tag{2.1}$$

Clearly, for any $\lambda > 0$, we have, using (2.1),

$$\begin{aligned} \omega^{2r}(f, \lambda t)_p \le C_1 K_{2r}(f, \widetilde{\Delta}, (\lambda t)^{2r})_p &\le C_1 \max\{1, \lambda^{2r}\} K_{2r}(f, \widetilde{\Delta}, t^{2r})_p \\ &\le C_2 \max\{1, \lambda^{2r}\} \omega^{2r}(f,t)_p \end{aligned} \tag{2.2}$$

where C_1 and C_2 are independent of t and λ.

Spherical polynomials of order k on $\mathbb{S}^{n-1}$ is defined by (see [7])

$$P_k(x) := \sum_{j=0}^{k} H_j(x)$$

where $H_j(x)$ is the spherical harmonic of order j, the trace of some homogeneous polynomials on $\mathbb{R}^n$

$$Q_j(x) := \sum_{i_1+i_2+\cdots+i_n=m} x_1^{i_1}\cdots x_n^{i_n}$$

of order j, where $x_i (i = 1, 2, \ldots, n)$ is the i-th coordinate of $x \in \mathbb{R}^n$. We denote by Π_k^n the collection of all spherical polynomials on $\mathbb{S}^{n-1}$ of degree no more than k.

The known Jackson operators are defined by (see [7])

$$J_{k,s}(f)(x) := \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} f(y)\mathscr{D}_{k,s}(\arccos x\cdot y)d\omega(y), \quad f \in L^p(\mathbb{S}^{n-1})$$

where k and s are positive integers, $d\omega(y)$ is the elementary surface piece of $\mathbb{S}^{n-1}$,

$$\mathscr{D}_{k,s}(\theta) := A_{k,s}^{-1}\left(\frac{\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2s}$$

is the classical Jackson kernel where $A_{k,s}^{-1}$ is a constant connected with k and s such that

$$\int_0^{\pi} \mathscr{D}_{k,s}(\theta)\sin^{2\lambda}\theta\, d\theta = 1, \quad \lambda = \frac{n-2}{2}.$$

We observe that

$$
\begin{aligned}
J_{k,s}(f)(x) &= \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} f(y)\mathscr{D}_{k,s}(\arccos x\cdot y)d\omega(y) \\
&= \int_0^{\pi}\left(\frac{1}{|\mathbb{S}^{n-2}|\sin^{2\lambda}\theta}\int_{x\cdot y=\cos\theta} f(y)d\omega'(y)\right)\mathscr{D}_{k,s}(\theta)\sin^{2\lambda}\theta\, d\theta \\
&= \int_0^{\pi} S_\theta(f)(x)\mathscr{D}_{k,s}(\theta)\sin^{2\lambda}\theta\, d\theta. \qquad (2.3)
\end{aligned}
$$

We introduce the definition of r-th Boolean sums of Jackson operator as follows (see [7]).

Definition 2.1 *The r-th ($r \geq 1$) Boolean sum of Jackson operator of degree k ($k \geq 1$) on $\mathbb{S}^{n-1}$ is defined by*

$$
\oplus^r J_{k,s}(f) := \left(-(I - J_{k,s})^r + I\right)(f), \quad f \in L^p(\mathbb{S}^{n-1}), \qquad (2.4)
$$

where s is a positive integer.

It is clear that

$$
\oplus^r J_{k,s}(f) = -\sum_{i=1}^{r}(-1)^i\binom{r}{i}(J_{k,s})^i(f). \qquad (2.5)
$$

We now make a brief introduction of projection operators $Y_j(\cdot)$ by ultraspherical (Gegenbauer) polynomials $\{G_j^\lambda\}_{j=1}^\infty$ ($\lambda = \frac{n-2}{2}$) for discussion of saturation property of $\oplus^r J_{k,s}$.

Ultraspherical polynomials $\{G_j^\lambda\}_{j=1}^\infty$ are defined in terms of the generating function (see [12]):

$$
\frac{1}{(1-2tr+r^2)^\lambda} = \sum_{j=0}^{\infty} G_j^\lambda(t) r^j
$$

where $|r| < 1$, $|t| \leq 1$. For any $\lambda > 0$, we have (see [12])

$$
G_1^\lambda(t) = 2\lambda t \qquad (2.6)
$$

and

$$
\frac{d}{dt}G_j^\lambda(t) = 2\lambda G_{j-1}^{\lambda+1}(t). \qquad (2.7)
$$

When $\lambda = \frac{n-2}{2}$ (see [**?**]),

$$
G_j^\lambda(t) = \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}P_j^n(t), \quad j = 0, 1, 2, \ldots
$$

where $P_j^n(t)$ is the Legendre polynomial of degree j (see [9]). Particularly,

$$
G_j^\lambda(1) = \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}P_j^n(1) = \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}, \quad j = 0, 1, 2, \ldots,
$$

therefore,

$$P_j^n(t) = \frac{G_j^\lambda(t)}{G_j^\lambda(1)}. \tag{2.8}$$

Besides, for any $j = 0, 1, 2, \ldots$, and $|t| \le 1$, $|P_j^n(t)| \le 1$ (see [9]).

The projection operators are defined by

$$Y_j(f)(x) := \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{\frac{n}{2}}} \int_{\mathbb{S}^{n-1}} G_j^\lambda(x \cdot y) f(y)\, d\omega(y).$$

It follows from (2.6), (2.7) and (2.8) that

$$\lim_{t \to 1-} \frac{1 - P_j^n(t)}{1 - P_1^n(t)} = \frac{j(j+2\lambda)}{2\lambda + 1}, \quad j = 0, 1, 2, \ldots. \tag{2.9}$$

Finally, we introduce the definition of saturation for operators (see [1]).

Definition 2.2 *Let $\varphi(\rho)$ be a positive function with respect to ρ, $0 < \rho < \infty$, tending monotonely to zero as $\rho \to \infty$. For a sequence of operators $\{I_\rho\}_{\rho > 0}$ if there exists $\mathcal{K} \subsetneqq L^p(\mathbb{S}^{n-1})$ such that*

(i) *If $\|I_\rho(f) - f\|_p = o(\varphi(\rho))$, then $I_\rho(f) = f$;*

(ii) *$\|I_\rho(f) - f\|_p = O(\varphi(\rho))$ if and only if $f \in \mathcal{K}$;*

then I_ρ is said to be saturated on $L^p(\mathbb{S}^{n-1})$ with order $O(\varphi(\rho))$ and $\mathcal{K}$ is called its saturation class.

3 Some Lemmas

In this section, we show some lemmas as the preparation for the proof of the main results.

Lemma 3.1 *For any $f \in L^p(\mathbb{S}^{n-1})$ and any positive integers k, r, u, s, we have,*

(i) *$\oplus^u J_{k,s}(f)$ is a spherical polynomial of degree no more than $s(k-1)$, which implies $\widetilde{\Delta}^r(\oplus^u J_{k,s}(f)) \in L^p(\mathbb{S}^{n-1})$;*

(ii) *$\| \oplus^u J_{k,s}(f)\|_p \le 2^u \|f\|_p$;*

(iii) *$\|\widetilde{\Delta}^r(\oplus^u J_{k,s}(f))\|_p \le C k^{2r} \|f\|_p$;*

(iv) *If $\widetilde{\Delta}^r g \in L^p(\mathbb{S}^{n-1})$, then $\|\widetilde{\Delta}^r(\oplus^u J_{k,s}(g))\|_p \le 2^u \|\widetilde{\Delta}^r g\|_p$.*

Proof. (i) Since $\mathscr{D}_{k,s}(\theta)$ is an even trigonometric polynomial of degree no more than $k(s-1)$, then $J_{k,s}(f)$ is a spherical polynomial of degree no more than $s(k-1)$. Thus we can prove by induction that $\oplus^u J_{k,s}(f)$ is a spherical polynomial of degree no more than $s(k-1)$.

(ii) Using the contraction of translation operator that (see for instance [**?**])

$$\|S_\theta(f)\|_p \le \|f\|_p, \quad 0 < \theta < \pi \tag{3.10}$$

as well as (2.3), we have

$$\|J_{k,s}(f)\|_p \ \le \ \int_0^\pi \|S_\theta(f)\|_p \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta\, d\theta \le \|f\|_p.$$

Using (2.5), we get

$$\| \oplus^u J_{k,s}(f)\|_p \quad \leq \quad 2^u \, \|f\|_p.$$

(iii) Using Bernstein-inequality on the sphere (see [10]), that is,

$$\|\widetilde{\Delta} P_k\|_p \leq Ck^2 \|P_k\|_p, \quad P_k \in \Pi_k^n,$$

we may easily obtain by induction that

$$\|\widetilde{\Delta}^r P_k\|_p \leq C^r k^{2r} \|P_k\|_p.$$

Noting that (i) implies $\oplus^u J_{k,s}(f) \in \Pi^d_{s(k-1)}$, we thus have

$$\|\widetilde{\Delta}^r(\oplus^u J_{k,s}(f))\|_p \quad \leq \quad Ck^{2r}\| \oplus^u J_{k,s}(f)\|_p = C_r k^{2r}\|f\|_p,$$

where C and C_r are independent of f and k.

(iv) The fact that $\widetilde{\Delta} S_\theta(g) = S_\theta \widetilde{\Delta}(g)$ (see [7]) implies

$$\widetilde{\Delta}^r(\oplus^u J_{k,s}(g)) = \oplus^u J_{k,s}(\widetilde{\Delta}^r g). \tag{3.11}$$

We thus use (ii) and find

$$\|\widetilde{\Delta}^r(\oplus^u J_{k,s}(g))\|_p = \| \oplus^u J_{k,s}(\widetilde{\Delta}^r g)\|_p \leq 2^u \|\widetilde{\Delta}^r g\|_p.$$

This completes the proof of Lemma 3.1. $\square$

Lemma 3.2 *For $\beta \geq -1$, $2s \geq \beta + n - 2$, $0 < \gamma \leq \pi$, and $n \geq 3$, we have*

$$\int_0^\gamma \theta^\beta \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta \, d\theta \approx k^{-\beta}, \tag{3.12}$$

where $\lambda = \frac{n-2}{2}$, and s, n, k are positive integers.

Proof. A simple computation gives

$$\int_0^\gamma \theta^\beta \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta d\theta = \frac{\displaystyle\int_0^\gamma \theta^\beta \left(\frac{\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2s} \sin^{2\lambda}\theta d\theta}{\displaystyle\int_0^\gamma \left(\frac{\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}\right)^{2s} \sin^{2\lambda}\theta d\theta} \quad \approx \quad k^{-\beta}.$$

The proof of Lemma 3.2 is completed. $\square$

The following lemma is an improved version of Lemma 2.1 of [14], which is useful for the proof of Bernstein-type inequality.

Lemma 3.3 *Assuming that $\{\sigma_v\}_{v=1}^\infty, \{\tau_v\}_{v=1}^\infty$ are nonnegative sequences with $\sigma_1 = 0$. For $a > 1$ and positive integer k, if the inequalities*

$$\sigma_k \leq \left(\frac{av}{k}\right)^p \sigma_v + \tau_v, \quad v = 1, 2, \ldots, k$$

hold, then there exists $0 < b_m < 1$, $b_m \to 1$, as $m \to \infty$, such that

$$\sigma_k \leq C_m k^{-b_m p} \sum_{v=1}^k v^{b_m p - 1} \tau_v.$$

Proof. First we take $m > 1$, such that $\ln \dfrac{m}{a} > 1$. Then there exists positive integer N, such that $m^N \le k < m^{N+1}$. Take $s_v \ge 0$ such that $km^{-v-1} < s_v \le km^{-v}$, $v = 0, 1, \ldots, N$ as well as

$$\tau_{s_v} \le \tau_j \quad (km^{-v-1} < j \le km^{-v}).$$

Set $s_{N+1} = 1$, $b_m = \dfrac{\ln \frac{m}{a}}{\ln m} < 1$. Clearly $b_m \to 1$, as $m \to \infty$. Then

$$\begin{aligned}
\sigma_k &\le \left(\frac{as_0}{k}\right)^p \sigma_{s_0} + \tau_{s_0} \\
&\le k^{-p} \sum_{v=0}^{N} (a^{v+1} s_v)^p \left(\sigma_{s_v} - \left(\frac{as_{v+1}}{s_v}\right)^p \sigma_{s_{v+1}} \right) + \tau_{s_0} \\
&\le m^p \sum_{v=0}^{N} \left(\frac{m}{a}\right)^{-p(v+1)} \tau_{s_{v+1}} + \tau_{s_0} \\
&\le m^p \sum_{v=0}^{N+1} m^{-\left(\frac{\ln \frac{m}{a}}{\ln m}\right) pv} \tau_{s_v} \\
&\le C_m k^{-b_m p} \left(\sum_{v=0}^{N+1} \sum_{km^{-v-1} < j \le km^{-v}} j^{b_m p - 1} \tau_j + \tau_1 \right) \\
&= C_m k^{-b_m p} \sum_{j=1}^{k} j^{b_m p - 1} \tau_j.
\end{aligned}$$

This finishes the proof of Lemma 3.3. □

The following lemma gives the description of $\oplus^r J_{k,s}$ by multipliers, which was proved in [7].

Lemma 3.4 *For $f \in L^p(\mathbb{S}^{n-1})$, there holds*

$$\oplus^r J_{k,s}(f) \;=\; \sum_{j=0}^{\infty} {}^r\xi_{k,s}(j) Y_j(f)$$

where

$${}^r\xi_{k,s}(j) = 1 - \left(\int_0^{\gamma} \mathscr{D}_{k,s}(\theta) \left(1 - P_j^n(\cos\theta)\right) \sin^{2\lambda}\theta \, d\theta \right)^r, \quad j = 0, 1, 2, \ldots$$

and the convergence of the series is meant in a weak sense.

The final lemma is useful for determining the saturation order. It can be deduced by the methods in [1] and [3].

Lemma 3.5 *Suppose that $\{I_\rho\}_{\rho>0}$ is a sequence of operators on $L^p(\mathbb{S}^{n-1})$, and there exists series $\{\lambda_\rho(j)\}_{j=1}^{\infty}$ with respect to ρ, such that*

$$I_\rho(f)(x) = \sum_{j=0}^{\infty} \lambda_\rho(j) Y_j(f)(x)$$

for every $f \in L^p(\mathbb{S}^{n-1})$. If for any $j = 0, 1, 2, \ldots,$ there exists $\varphi(\rho) \to 0+$ $(\rho \to \rho_0)$ such that

$$\lim_{\rho\to\rho_0} \frac{1-\lambda_\rho(j)}{\varphi(\rho)} = \tau_j \neq 0,$$

then $\{I_\rho\}_{\rho>0}$ is saturated on $L^p(\mathbb{S}^{n-1})$ with the order $O(\varphi(\rho))$ and the collection of all constants is the invariant class for $\{I_\rho\}_{\rho>0}$ on $L^p(\mathbb{S}^{n-1})$. $\square$

4 Main Results and Their Proof

In this section, we shall state and prove the main results, that is, the lower and upper bounds as well as the saturation order for Boolean sums of Jackson operators on $L^p(\mathbb{S}^{n-1})$.

Theorem 4.1 *Let $2s \geq n$, and let $\{\oplus^r J_{k,s}\}_{k=1}^{\infty}$ be the sequence of Boolean sums of Jackson operators defined above. Then for any positive integers k and r as well as sufficiently smoothing $g \in L^p(\mathbb{S}^{n-1}), 1 \leq p \leq \infty$ such that $\widetilde{\Delta}^r g \in L^p(\mathbb{S}^{n-1})$, we have*

$$\| \oplus^r J_{k,s}(g) - g\|_p \leq C_1 k^{-2r} \|\widetilde{\Delta}^r g\|_p, \tag{4.13}$$

therefore, for any $f \in L^p(\mathbb{S}^{n-1})$, we have

$$\| \oplus^r J_{k,s}(f) - f\|_p \leq C_2 \omega^{2r}(f, k^{-1})_p, \tag{4.14}$$

where C_1 and C_2 are constants independent of f and k.

Proof. By Definition 2.1, we have

$$\oplus^r J_{k,s}(g)(x) - g(x) = -(I - J_{k,s})^r (g)(x).$$

Now we prove (4.13) by induction. For $r = 1$,

$$S_\theta(g)(x) - g(x) = \int_0^\theta \sin^{-2\lambda}\tau \int_0^\tau \sin^{2\lambda} u S_u(\widetilde{\Delta} g)(x) du d\tau$$

(see [10]) implies (explained below)

$$\begin{aligned}
\|J_{k,s}(g) - g\|_p &= \left\| \int_0^\pi D_{k,s}(\theta) \left(S_\theta(g)(\cdot) - g(\cdot)\right) \sin^{2\lambda}\theta d\theta \right\|_p \\
&\leq \int_0^\pi D_{k,s}(\theta) \sin^{2\lambda}\theta \int_0^\theta \sin^{-2\lambda}\tau \int_0^\tau \sin^{2\lambda} u \left\| S_u(\widetilde{\Delta} g)\right\|_p du d\tau d\theta \\
&\leq \sup_{\theta>0} \left\{ \theta^{-2} \int_0^\theta \sin^{-2\lambda}\tau \int_0^\tau \sin^{-2\lambda} u du\, d\tau \right\} \left(\int_0^\pi \theta^2 D_{k,s}(\theta) \sin^{2\lambda}\theta d\theta \right) \|\widetilde{\Delta} g\|_p \\
&\leq C k^{-2} \|\widetilde{\Delta} g\|_p,
\end{aligned} \tag{4.15}$$

where the Minkowski inequality is used in the first inequality, the second one by (3.10) and the third one is deduced from Lemma 3.2.

Assume that for any fixed positive integer u,

$$\| \oplus^u J_{k,s}(g) - g\|_p \leq Ck^{-2u}\|\widetilde{\Delta}^u g\|_p.$$

Then

$$\begin{aligned}\| \oplus^{u+1} J_{k,s}(g) - g\|_p &= \|(J_{k,s} - I)(\oplus^u J_{k,s}(g) - g)\|_p \leq Ck^{-2}\|\widetilde{\Delta}(\oplus^u J_{k,s}(g) - g)\|_p \\ &= Ck^{-2}\| \oplus^u J_{k,s}(\widetilde{\Delta}g) - \widetilde{\Delta}g\|_p \leq Ck^{-2u-2}\|\widetilde{\Delta}^{u+1}g\|_p,\end{aligned}$$

where the first inequality is by (4.15), the second one by (3.11), the last by induction assumption. Therefore, (4.13) holds.

Using (2.1) and noticing that $\oplus^u J_{k,s}$ is a linear operator, we obtain (4.14). This completes the proof of the theorem. □

Next, we establish an inverse inequality of strong type for $\oplus^r J_{k,s}$ on $L^p(\mathbb{S}^{n-1})$.

Theorem 4.2 *For positive $r \geq 1$ and $f \in L^p(\mathbb{S}^{n-1})$, $1 \leq p \leq \infty$, there exists a constant C independent of f and k such that*

$$\omega^{2r}(f, k^{-1})_p \leq C \max_{v \geq k} \| \oplus^r J_{v,s}(f) - f\|_p. \tag{4.16}$$

Proof. We first establish a Steckin-Marchaud type inequality, that is, for $f \in L^p(\mathbb{S}^{n-1})$,

$$\omega^{2r}(f, k^{-1})_p \leq C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r-1}\| \oplus^r J_{v,s}(f) - f\|_p,$$

where $0 < b_m < 1$, $b_m \to 1$, as $m \to \infty$.

Set

$$\sigma_v = v^{-2r}\|\widetilde{\Delta}^r \left(\oplus^r J_{v,s}(f)\right) \|_p, \quad \tau_v = \| \oplus^r J_{v,s}(f) - f\|_p, \quad v \geq 1.$$

Using Lemma 3.1, we have

$$\begin{aligned}\sigma_k &\leq k^{-2r}\|\widetilde{\Delta}^r \left(\oplus^r J_{k,s}(\oplus^r J_{v,s}(f))\right) \|_p + k^{-2r}\|\widetilde{\Delta}^r \left(\oplus^r J_{k,s}(\oplus^r J_{v,s}(f) - f)\right) \|_p \\ &\leq 2^r \left(\frac{v}{k}\right)^{2r} \left(v^{-2r}\|\widetilde{\Delta}^r \left(\oplus^r J_{v,s}(f)\right) \|_p\right) + C \| \oplus^r J_{v,s}(f) - f\|_p \\ &= \left(\frac{\sqrt{2}\, v}{k}\right)^{2r} \sigma_v + C\tau_v.\end{aligned}$$

By Lemma 3.3, we have

$$\sigma_k \leq C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r-1}\tau_v$$

for some large enough m.
That is,

$$k^{-2r}\|\widetilde{\Delta}^r \left(\oplus^u J_{k,s}(f)\right) \|_p \leq C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r-1}\| \oplus^r J_{v,s}(f) - f\|_p.$$

For $k \geq 1$, there exists a positive integer k_0, $\dfrac{k}{2} \leq k_0 \leq k$, such that

$$\| \oplus^r J_{k_0,s}(f) - f\|_p \leq \| \oplus^r J_{v,s}(f) - f\|_p, \quad \frac{k}{2} \leq v \leq k.$$

Thus

$$\begin{aligned} K_{2r}(f, \widetilde{\Delta}, k^{-2r})_p &\leq \| \oplus^r J_{k_0,s}(f) - f\|_p + k^{-2r} \|\widetilde{\Delta}^r \left(\oplus^r J_{k_0,s}(f)\right) \|_p \\ &\leq 2^{2r} k^{-2r} \sum_{\frac{k}{2} \leq v \leq k} v^{2r-1} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\quad + C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r - 1} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\leq C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r - 1} \| \oplus^r J_{v,s}(f) - f\|_p. \end{aligned}$$

From (2.1) it follows that

$$\omega^{2r}(f, k^{-1})_p \leq C_m k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r - 1} \| \oplus^u J_{v,s}(f) - f\|_p.$$

To finish our proof, we need the following inequalities.

$$\begin{aligned} \omega^{2r}(f, k^{-1})_p &\approx \frac{1}{k^{2r}} \max_{1 \leq v \leq k} v^{2r} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\approx \frac{1}{k^{2r+\frac{1}{4}}} \max_{1 \leq v \leq k} v^{2r+\frac{1}{4}} \| \oplus^r J_{v,s}(f) - f\|_p. \end{aligned} \tag{4.17}$$

In the first place, we prove the former inequality of (4.17) (explained below).

$$\begin{aligned} \omega^{2r}(f, k^{-1})_p &\leq C_1 k^{-2b_m r} \sum_{v=1}^{k} v^{2b_m r - 1} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\leq C_1 \left(k^{-2b_m r} \sum_{v=1}^{k} v^{-2(1-b_m)r-1} \right) \max_{1 \leq v \leq k} v^{2r} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\leq C_2 k^{-2r} \max_{1 \leq v \leq k} v^{2r} \| \oplus^r J_{v,s}(f) - f\|_p \\ &\leq C_3 k^{-2r} \max_{1 \leq v \leq k} v^{2r} \omega^{2r}(f, v^{-1})_p \\ &\leq C_4 \left(k^{-2r} \max_{1 \leq v \leq k} v^{2r} \left(\frac{k}{v}\right)^{2r} \right) \omega^{2r}(f, k^{-1})_p \\ &\leq C_4 \omega^{2r}(f, k^{-1})_p, \end{aligned}$$

where the fourth inequality is deduced by Theorem 4.1 and the fifth is by (2.2).
Thus

$$\omega^{2r}(f, k^{-1})_p \approx \frac{1}{k^{2r}} \max_{1 \leq v \leq k} v^{2r} \| \oplus^r J_{v,s}(f) - f\|_p.$$

In the same way, we have

$$\begin{aligned}
\omega^{2r}(f,k^{-1})_p &\leq C_1 k^{-2b_m r}\sum_{v=1}^{k} v^{2b_m r-1}\|\oplus^r J_{v,s}(f)-f\|_p \\
&\leq C_1\left(k^{-2b_m r}\sum_{v=1}^{k} v^{-2(1-b_m)r-\frac{1}{4}-1}\right)\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p \\
&\leq C_5 k^{-2r-\frac{1}{4}}\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p \\
&\leq C_6 k^{-2r-\frac{1}{4}}\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\omega^{2r}(f,v^{-1})_p \\
&\leq C_6 k^{-2r-\frac{1}{4}}\left(\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\left(\frac{k}{v}\right)^{2r}\right)\omega^{2r}(f,k^{-1})_p \\
&\leq C_7\omega^{2r}(f,k^{-1})_p,
\end{aligned}$$

that is,

$$\omega^{2r}(f,k^{-1})_p \approx \frac{1}{k^{2r+\frac{1}{4}}}\max_{v\geq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p.$$

Therefore

$$\begin{aligned}
\omega^{2r}(f,k^{-1})_p &\approx \frac{1}{k^{2r}}\max_{v\geq k} v^{2r}\|\oplus^r J_{v,s}(f)-f\|_p \\
&\approx \frac{1}{k^{2r+\frac{1}{4}}}\max_{v\geq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p. \qquad (4.18)
\end{aligned}$$

Now we can complete the proof of (4.16). Clearly, there exists $1\leq k_1\leq k$ such that

$$k_1^{2r+\frac{1}{4}}\|\oplus^r J_{k_1,s}(f)-f\|_p = \max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p.$$

Then it is deduced from (4.18) that

$$\begin{aligned}
k^{-2r}k_1^{2r}\|\oplus^r J_{k_1,s}(f)-f\|_p &\leq \frac{1}{k^{2r}}\max_{1\leq v\leq k} v^{2r}\|\oplus^r J_{v,s}(f)-f\|_p \\
&\leq C_8\frac{1}{k^{2r+\frac{1}{4}}}\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p \\
&= C_8 k^{-2r-\frac{1}{4}}k_1^{2r+\frac{1}{4}}\|\oplus^r J_{k_1,s}(f)-f\|_p.
\end{aligned}$$

This implies $k_1\approx k$. Applying (4.18) again implies

$$\begin{aligned}
\omega^{2r}(f,k^{-1})_p &\leq C_5\frac{1}{k^{2r+\frac{1}{4}}}\max_{1\leq v\leq k} v^{2r+\frac{1}{4}}\|\oplus^r J_{v,s}(f)-f\|_p \\
&= C_5\frac{1}{k^{2r+\frac{1}{4}}}(k_1^{2r+\frac{1}{4}}\|\oplus^r J_{k_1,s}(f)-f\|_p) \\
&\leq C_5\max_{k_1\leq v\leq k}\|\oplus^r J_{v,s}(f)-f\|_p.
\end{aligned}$$

Noticing that $k_1 \approx k$, we may rewrite the above inequality as

$$\omega^{2r}(f, k^{-1})_p \le C \max_{v \ge k} \| \oplus^r J_{v,s}(f) - f\|_p.$$

This completes the proof of Theorem 4.2. $\Box$

Theorem 4.3 *$\{\oplus^r J_{k,s}\}_{k=1}^{\infty}$ are saturated on $L^p(\mathbb{S}^{n-1})$ with order k^{-2r} and the collection of constants is their invariant class.*

Proof. We first prove for $j = 0, 1, 2, \dots,$

$$\lim_{k\to\infty} \frac{1 - {}^1\xi_k(j)}{1 - {}^1\xi_k(1)} \quad = \quad \frac{j(j+2\lambda)}{2\lambda+1}. \tag{4.19}$$

In fact, for any $0 < \delta < \pi$, it follows from (3.12) that

$$\begin{aligned} \int_\delta^\pi \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta \, d\theta \quad &\le \quad \int_\delta^\pi \left(\frac{\theta}{\delta}\right)^3 \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta \, d\theta \\ &\le \quad \delta^{-3} \int_0^\pi \theta^3 \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta \, d\theta \;\le\; C_{\delta,s} k^{-3}. \end{aligned}$$

For $v = 1, 2, \dots,$ we have, using (3.12) again,

$$\begin{aligned} 1 - {}^1\xi_k(1) = \int_0^\pi \mathscr{D}_{k,s}(\theta) \left(1 - \frac{G_1^\lambda(\cos\theta)}{G_1^\lambda(1)}\right) \sin^{2\lambda}\theta \, d\theta \quad &\approx \quad \int_0^\pi \mathscr{D}_{k,s}(\theta) \sin^2\frac{\theta}{2} \sin^{2\lambda}\theta \, d\theta \\ &\approx \quad k^{-2}. \end{aligned} \tag{4.20}$$

We deduce from (2.9) that for any $\epsilon > 0$, there exists $\delta > 0$, for $0 < \theta < \delta$, such that

$$\left| \big(1 - P_j^n(\cos\theta)\big) - \frac{j(j+2\lambda)}{2\lambda+1}\left(1 - P_1^n(\cos\theta)\right) \right| \quad \le \quad \epsilon \left(1 - P_1^n(\cos\theta)\right).$$

Then it follows that

$$\begin{aligned} & \left| \big(1 - {}^1\xi_k(j)\big) - \frac{j(j+2\lambda)}{2\lambda+1}\left(1 - {}^1\xi_k(1)\right) \right| \\ = \quad & \left| \int_0^\pi \mathscr{D}_{k,s}(\theta) \big(1 - P_j^n(\cos\theta)\big) \sin^{2\lambda}\theta \, d\theta \right. \\ & \left. - \int_0^\pi \mathscr{D}_{k,s}(\theta) \left(1 - P_1^n(\cos\theta)\right) \frac{j(j+2\lambda)}{2\lambda+1} \sin^{2\lambda}\theta \, d\theta \right| \\ = \quad & \left| \int_0^\pi \mathscr{D}_{k,s}(\theta) \left(\big(1 - P_j^n(\cos\theta)\big) - \big(1 - P_1^n(\cos\theta)\big) \frac{j(j+2\lambda)}{2\lambda+1} \right) \sin^{2\lambda}\theta \, d\theta \right| \\ \le \quad & \int_0^\delta \mathscr{D}_{k,s}(\theta)\, \epsilon\, \left(1 - P_1^n(\cos\theta)\right) \sin^{2\lambda}\theta \, d\theta + 2\int_\delta^\pi \mathscr{D}_{k,s}(\theta) \sin^{2\lambda}\theta \left(1 + \frac{j(j+2\lambda)}{2\lambda+1}\right) d\theta \\ \le \quad & C\epsilon\, k^{-2} + C_{\delta,s} k^{-3}. \end{aligned}$$

So, (4.19) holds. By Lemma 3.4,

$$\oplus^r J_{k,s}(f) - f = \sum_{j=0}^{\infty} (1 -^r \xi_{k,s}(j))\, Y_j(f)$$

and for $j = 1, 2, \ldots,$

$$1 - {}^r\xi_{k,s}(j) = \left(\int_0^{\gamma} \mathscr{D}_{k,s}(\theta) \left(1 - P_j^n(\cos\theta)\right) \sin^{2\lambda}\theta \, d\theta \right)^r = \left(1 - {}^1\xi_{k,s}(j)\right)^r.$$

Combining with (4.19) and (4.20), we have,

$$\lim_{k\to\infty} \frac{1 - {}^r\xi_{k,s}(j)}{1 - {}^r\xi_{k,s}(1)} = \left(\frac{j(j+2\lambda)}{2\lambda}\right)^r \neq 0$$

and $1 - {}^r\xi_{k,s}(1) \approx k^{-2r}$. Using of Lemma 3.5, we finish the proof of Theorem 4.3. $\square$

We obtain the following corollary from Theorem 4.1, Theorem 4.2 and Theorem 4.3.

Corollary 4.1 *For positive integers r and s, $2s \geq n$, $0 < \alpha \leq 2r$, $f \in L^p(\mathbb{S}^{n-1}), 1 \leq p \leq \infty$, and the sequence of Boolean sums of Jackson operators $\{\oplus^r J_{k,s}\}_{k=1}^{\infty}$ given by (2.4), the following statements are equivalent.*
(i) $\| \oplus^r J_{k,s}(f) - f\|_p = O(k^{-\alpha}) \quad (k \to \infty)$;
(ii) $\omega^{2r}(f, \delta)_p = O(\delta^{\alpha}) \quad (\delta \to 0)$. $\square$

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